

Göteborg-ITP-95-16
hep-th/9508124
August, 1995

LOW ENERGY DYNAMICS OF MONOPOLES IN $N=2$ SYM WITH MATTER

Martin Cederwall, Gabriele Ferretti, Bengt E.W. Nilsson and Per Salomonson

*Institute for Theoretical Physics
Chalmers University of Technology and Göteborg University
S-412 96 Göteborg, Sweden*

Abstract: We derive, for $N = 2$ super-Yang–Mills with gauge group $SU(2)$ and massless matter, the supersymmetric quantum mechanical models describing the time evolution of multi-monopole configurations in the low energy approximation. This is a first step towards identifying the solitonic states mapped to fundamental excitations by duality in the model with four hypermultiplets in the fundamental representation.

email addresses: `tfemc@fy.chalmers.se`
`ferretti@fy.chalmers.se`
`tfebn@fy.chalmers.se`
`tfeps@fy.chalmers.se`

1. INTRODUCTION

The holomorphicity appearing in supersymmetric gauge theories provides a unique opportunity to understand their non-perturbative structures. This has been amply demonstrated in a number of recent papers where asymptotically free (or ultraviolet finite) Yang–Mills theories with $N=1, 2$ or 4 supersymmetries coupled to a variety of matter multiplets have been shown to possess a number of interesting exact properties [1-4].

One of the most intriguing features that can now be investigated is the duality between strong and weak coupling. The possibility in field theory of such a property was first discussed by Montonen and Olive [5] and in the context of string theory by Font et al. [6].

For $N=1, 2$, duality relates different low energy descriptions of the same theory at different points in the quantum moduli space. While in these cases duality is not a symmetry of the theory, the situation changes drastically if we instead consider theories with vanishing β -function, e.g. $N=4$ super-Yang–Mills. This theory is known to be ultraviolet finite at the perturbative [7-12] as well as non-perturbative [13] level. Here one can perform a duality transformation from weak to strong coupling and then tune to weak coupling thereby making a comparison possible. This theory was conjectured to be selfdual in ref. [14].

As emphasized recently by Seiberg and Witten [2], there is another candidate, perhaps more interesting, for such a selfdual theory, namely $SU(2)$ $N=2$ super-Yang–Mills coupled to four hypermultiplets in the fundamental representation of the gauge group. Such $N=2$ theories are perturbatively finite provided the number of hypermultiplets N_f and the gauge group $SU(N_c)$ are related by $N_f=2N_c$. It was argued in [2] that this is most likely true also non-perturbatively. The purpose of the present paper is to derive the quantum mechanical models governing the dynamics of the zero-modes obtained from the $N_c=2$ theory with hypermultiplets in the background of a BPS (multi-)monopole [15,16].

The required steps needed to obtain the lagrangian for the zero-modes are well-known and can be found in many places in the literature; see e.g. [17,18,19]. Here we will use a somewhat unusual formulation of the $N=2$ theory based on quaternions. There are several reasons why quaternions are convenient in this context. One is that the lagrangian in $D=(1+3)$ dimensions is most easily derived from the $N=1$, $D=(1+5)$ analogue, and that this theory has a nice formulation in terms of quaternions. Just as the isomorphism between $Spin(1,3)$ and $Sl(2, \mathbb{C})$ makes it possible to introduce the extremely useful van der Weerden notation (see e.g. the book by Wess and Bagger [20]), in $D=(1+5)$ the fact that $Spin(1,5)$ and $Sl(2, \mathbb{H})$ are isomorphic [21,22] leads to a two-component formulation quite similar to the one in $D=(1+3)$ but where the “Pauli matrices” are now five in number due to their entries being quaternionic. This is explained in section 2. A second reason is that many properties of monopoles become particularly apparent if discussed in terms of three dimensional space accompanied by a fourth euclidean dimension as e.g. in the context of the Atiyah-Singer universal bundle [23]. This extra dimension is just a fourth space dimension of the $(1+5)$ -dimensional theory we start from, and as we will see there is actually no need for giving the theory in $(1+3)$ dimensions at all.

Quaternions are in fact already widely used in connection with monopoles but for a reason slightly different from ours. When seeking explicit functional forms of the zero modes and the monopole solutions themselves it has in the past turned out to be very fruitful to conduct the search in the language of quaternions [24].

In section 3 we present the quaternionic formulation in $D=(1+5)$ of the relevant super-

Yang–Mills theory coupled to hypermultiplets, while section 4 is devoted to the derivation of the zero-mode formulas needed to integrate the lagrangian over three-space. We give e.g. exact expressions for the curvature of moduli space as well as the curvature of the relevant index bundle that appears due to the presence of new fermionic zero-modes coming from the hypermultiplets.

The supersymmetry of the quantum mechanical model for the zero-modes is discussed in detail in section 5. The lagrangian found here coincides with the one obtained in section 6 by integrating the field theory lagrangian of section 3 over space using the monopole formulas in section 4. Our results are consistent with the ones stated by Blum [25] for the $N=4$ theory and extends the ones of Gauntlett [26] for $N=2$ by including hypermultiplets. We plan to continue these investigations in a future publication. In section 7, we outline the way to proceed in order to find the necessary forms on moduli space that would vindicate the conjecture about selfduality of this theory along the lines already used by Sen [27] for the $N=4$ theory.

2. QUATERNIONIC NOTATION

Even though the concept of quaternions is well known among physicists, not many are accustomed to actually using them in calculations. This preliminary section may be skipped over by the reader already familiar with the use of quaternions. The reason we choose a quaternionic notation throughout this paper is twofold: firstly, it provides a natural and compact formulation of supersymmetric Yang–Mills theory in six dimensions, the easiest way to $N=2$ in four dimensions, secondly, they arise naturally in the context of monopoles and their moduli. Using a quaternionic notation in $D=(1+5)$ enables us to go directly to a most elegant version of monopole theory, formulated on a euclidean four-dimensional space, without explicitly considering $(1+3)$ -dimensional Minkowski space.

A quaternion $h \in \mathbb{H}$ is a collection of four real numbers h_a , $a = 1, \dots, 4$, arranged as $h = h_a e_a$, where $e_4 = 1$, $e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k$, $i, j = 1, 2, 3$. Under quaternionic conjugation, denoted by $*$, $h \rightarrow h^* = h_4 - h_i e_i$. Conjugation is an anti-involution, i.e. $(hh')^* = h'^* h^*$. We denote the real part by $\text{Re } h = h_4 = \frac{1}{2}(h + h^*)$. The norm of a quaternion is given by $|h|^2 = hh^* = h_a h_a$. The unit quaternions form the group $SU(2)$.

In four euclidean dimensions, a quaternion can represent a vector or any of the two spinor chiralities. The transformation rules for vectors v and spinors s and c under the rotation group $SU(2) \times SU(2)$ are:

$$\begin{aligned} v &\rightarrow h v g^* , \\ s &\rightarrow h s , \\ c &\rightarrow g c , \end{aligned} \tag{2.1}$$

where h and g are unit norm quaternions. The group of three-dimensional rotations is the diagonal subgroup. These transformations have been chosen in order to satisfy a triality, formally identical to the one for the three octonionic representations of $Spin(8)$, i.e. “ $v^* s = c$ ” and cyclic under $v \rightarrow s^* \rightarrow c \rightarrow v$. A scalar is formed as $\text{Re}(v' v^*)$ or similarly for the spinors (i.e. the norm is covariant). Note that there is room for yet another $SU(2)$ transforming s and c from the right. Such “extra” symmetries will appear as global symmetries in the field theory.

Gamma matrices in four euclidean dimensions are

$$\Gamma_\mu = \begin{bmatrix} 0 & e_\mu \\ e_\mu^* & 0 \end{bmatrix} , \tag{2.2}$$

so the Weyl equations for the two spinor chiralities may be written

$$\begin{aligned} D^*s &= 0 , \\ Dc &= 0 , \end{aligned} \tag{2.3}$$

where $D = e_\mu D_\mu$. We also note that quaternions $a = vv'^*$ and $b = v^*v'$ (and also corresponding expressions with spinors) have well-defined transformation properties. Their real parts are scalars and their imaginary parts contain a selfdual and an anti-selfdual anti-symmetric tensor, respectively. An example to be used is the covariant equation $D^*v=0$, that in components reads (antisymmetrization always has weight one)

$$\begin{aligned} D_\mu v_\mu &= 0 , \\ D_{[\mu} v_{\nu]} - \frac{1}{2} \varepsilon_{\mu\nu\kappa\lambda} D_\kappa v_\lambda &= 0 . \end{aligned} \tag{2.4}$$

Concerning $(1+5)$ -dimensional Minkowski space, there is a two-component spinor notation relying on the isomorphism $Spin(1,5) \approx Sl(2; \mathbb{H})$ [21,22]. The formalism is quite analogous to the $Sl(2; \mathbb{C})$ spinor notation in $D=(1+3)$. A difference is that there is no “ $\varepsilon_{\alpha\beta}$ ”, so spinor indices can not be lowered and raised. Therefore, there are two inequivalent spinor chiralities. In terms of the euclidean spinors, a six-dimensional spinor (in any of the two chiralities) is

$$S = \begin{bmatrix} s \\ c \end{bmatrix} . \tag{2.5}$$

The six-dimensional sigma matrices are ($M=0, \dots, 5$)

$$\begin{aligned} \Sigma^0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \Sigma^\mu = \begin{bmatrix} 0 & e_\mu \\ e_\mu^* & 0 \end{bmatrix} , \quad \Sigma^5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \\ \tilde{\Sigma}^0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \tilde{\Sigma}^\mu = - \begin{bmatrix} 0 & e_\mu \\ e_\mu^* & 0 \end{bmatrix} , \quad \tilde{\Sigma}^5 = - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \end{aligned} \tag{2.6}$$

and Lorentz transformations act by left multiplication by quaternionic matrices $\Sigma^{MN} = \Sigma^{[M} \tilde{\Sigma}^{N]}$ or $\tilde{\Sigma}^{MN} = \tilde{\Sigma}^{[M} \Sigma^{N]}$. Again there is room for an extra commuting $SU(2)$ acting from the right. If one instead of reducing to four euclidean dimensions goes to $(1+3)$ -dimensional Minkowski space, the spinors are acted on by left multiplication by complex matrices generating $Sl(2; \mathbb{C})$. Thereby the quaternions are split into $\mathbb{C} \oplus \mathbb{C}$, so that S contains two four-dimensional Weyl spinors.

3. N=2 SUPER-YANG-MILLS WITH MATTER MULTIPLETS

The action for $D=(1+5)$ super-Yang-Mills with massless matter is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} \text{Re} (\lambda^\dagger \Sigma^M D_M \lambda) \\ &\quad - \frac{1}{2} D_M q_f^* D^M q_f + \frac{1}{2} \text{Re} (\psi_f^\dagger \tilde{\Sigma}^M D_M \psi_f) + \text{Re} (\psi_f^\dagger \lambda q_f^*) + \frac{1}{8} (q_f^* \times q_f)^2 . \end{aligned} \tag{3.1}$$

For the sake of compactness of notation, we suppress representation indices for the gauge group, which throughout this paper is $SU(2)$. Here, the gauge potential A_M and the fermion λ form

the vector multiplet, and transform in the adjoint representation of the gauge group. λ is a two-component quaternionic spinor of the type discussed above. The fields q and ψ form the matter hypermultiplets: the index f labels the multiplets, which each can transform in any representation of the gauge group. In practice, the only candidates are the adjoint and fundamental, if asymptotic freedom is to be retained. In the case of adjoint multiplets, the maximum number is then one, which gives the $N = 2$ theory in $D = (1+5)$, which can be dimensionally reduced to $N = 4$ in $D = (1+3)$. In the case of fundamental matter, $f = 1, \dots, N_f$, where $N_f \leq 4$. The upper limits give finite theories [7-13,2,28]. The matter bosons q are Lorentz scalar quaternions, and the fermions ψ are again two-component quaternionic spinors, of the opposite (six-dimensional) chirality compared to λ . The cross product in the last term denotes the formation of an element in the adjoint.

The supersymmetry transformations are:

$$\begin{aligned} \delta A_M &= \text{Re}(\varepsilon^\dagger \Sigma_M \lambda) , & \delta q_f &= \psi_f^\dagger \varepsilon , \\ \delta \lambda &= -\frac{1}{2} F_{MN} \tilde{\Sigma}^{MN} \varepsilon + \frac{1}{2} \varepsilon (q_f^* \times q_f) , & \delta \psi_f &= \Sigma^M \varepsilon D_M q_f^* . \end{aligned} \quad (3.2)$$

The parameter ε is in the same spinor representation as λ and transforms the same way under global symmetries (see below). The supersymmetry algebra closes modulo equations of motion for the fermions, since no auxiliary fields are present in (3.1).

There is a global $SU(2) \times SU(2)$ symmetry of this action, transforming the fields as

$$\begin{aligned} A_M &\rightarrow A_M , & q &\rightarrow g^* q h , \\ \lambda &\rightarrow \lambda h , & \psi &\rightarrow \psi g . \end{aligned} \quad (3.3)$$

The simplest way of dimensionally reducing to the $N = 2$ action in $D = (1+3)$ [29] is to keep the spinors quaternionic. Since Lorentz transformations only act with left multiplication by complex matrices, they effectively split into pairs of $Sl(2; \mathbb{C})$ spinors. Also, the vectors split into vectors and scalars. For our purposes, this is not convenient to do explicitly, since even though monopole physics takes place in four-dimensional Minkowski space, the best mathematical formulation makes use of a four-dimensional euclidean space, where a Higgs field is taken as the fourth component of the vector potential. This is achieved directly from the six-dimensional action by singling out the four euclidean dimensions labeled by μ .

4. MONOPOLE MODULI

Our aim is to identify the bosonic and fermionic moduli (zero-modes) around a multi-monopole solution, and write down an action for slow (adiabatic) motion of these moduli. A (static) multi-monopole solution is a gauge field configuration $A_\mu(x)$ where $\mu = 1, 2, 3, 4$, independent of x^4 , whose field strength is selfdual. The components A_0 and A_5 as well as all other fields vanish. The Higgs field A_4 takes a non-zero value at spatial infinity, breaking the gauge $SU(2)$ to $U(1)$. Half of the supersymmetry is broken by a monopole solution (see below). We will not go into great detail on monopole theory (see [30] and references therein), except for some information about their moduli spaces needed in this paper. The total moduli space decomposes into the disjoint union of manifolds of solutions with different (integer) magnetic charge k . All statements about dimensionalities of moduli spaces and spaces of solutions are made for positive k , and are the same for k negative. The notation is mostly a quaternionic version of the one in ref. [31]. The reader

should not confuse our use of quaternions with that in [24], where the $SU(2)$ gauge generators act with right quaternionic multiplication in the fundamental representation.

A tangent vector to the moduli space must satisfy

$$\begin{aligned} D_\mu \delta_m A_\mu &= 0 , \\ D_{[\mu} \delta_m A_{\nu]} - \frac{1}{2} \varepsilon_{\mu\nu\kappa\lambda} D_\kappa \delta_m A_\lambda &= 0 \end{aligned} \quad (4.1)$$

(the second equation stating that $F_{\mu\nu}$ remains selfdual, the first one making the tangent vectors orthogonal to gauge modes in the metric stated below), which can be written in quaternionic notation as (see eq. (2.4))

$$D^* \delta_m A = 0 , \quad (4.2)$$

where $\delta_m A = \delta_m A_\mu e_\mu$. We immediately recognize that if $\delta_m A$ is a tangent vector, then so is $\delta_m A h$ for any quaternion h . This quaternionic structure of the zero-mode equation (acting on the scalar and anti-selfdual components of (4.1)) is translated into a quaternionic structure acting on the tangent bundle of moduli space. The moduli space is hyper-Kähler, with the three complex structures induced as

$$J^{(i)}_m{}^n \delta_n A = \delta_m A e_i . \quad (4.3)$$

It is well known that the moduli space \mathcal{M}_k of k -monopole solutions (modulo gauge transformations) is a $4k$ -dimensional hyper-Kähler manifold.

A preferred metric is

$$g_{mn} = \int d^3x \operatorname{Tr} \delta_m A_\mu \delta_n A_\mu = \int d^3x \operatorname{Tr} \operatorname{Re} (\delta_m A^* \delta_n A) \equiv \langle \delta_m A, \delta_n A \rangle . \quad (4.4)$$

This metric is natural because it derives from the kinetic term of the gauge field, in the sense that slow motion on moduli space is geodesic with respect to it (see sections 5 and 6). One can in fact collect the metric and the complex structures in

$$\langle \delta_m A, \delta_n A h^* \rangle = h_4 g_{mn} + h_i J^{(i)}_m{}^p g_{pn} = h_a J^{(a)}_{mn} . \quad (4.5)$$

where h is a quaternion.

The tangent vectors can be written as

$$\delta_m A = \partial_m A - D \varepsilon_m , \quad (4.6)$$

where the gauge parameters ε_m are chosen so that the first equation in (4.1) is satisfied. The operator $s_m = \partial_m + \operatorname{ad} \varepsilon_m$ together with D_μ forms a covariant derivative on the “universal bundle” $\mathcal{M}_k \times \mathbb{R}^4$ [23]. It fulfills

$$[s_m, D] = \delta_m A \quad (4.7)$$

and Jacobi identities show among other things that

$$\phi_{mn} \equiv [s_m, s_n] = 2(D^* D)^{-1} \operatorname{Re} (\delta_m A^* \cdot \delta_n A) , \quad (4.8)$$

where the dot indicates the adjoint action in the gauge $SU(2)$.

The Christoffel connection of the metric (4.4) is

$$\Gamma^m_{np} = g^{mq} \int d^3x \operatorname{Tr} \delta_q A_\mu s_n \delta_p A_\mu = g^{mq} \int d^3x \operatorname{Tr} \operatorname{Re} (\delta_q A^* s_n \delta_p A) = g^{mq} \langle \delta_q A, s_n \delta_p A \rangle . \quad (4.9)$$

Using that zero-modes and higher modes together form a complete set, and that zero-modes are orthogonal to higher modes, the validity of this formula can be verified, as well as the fact that the complex structures are covariantly constant. When calculating the curvature tensor, one obtains

$$R_{mnpq} = \langle \delta_p A, \phi_{mn} \delta_q A \rangle + \langle s_m \delta_p A, \Pi_+ s_n \delta_q A \rangle - \langle s_n \delta_p A, \Pi_+ s_m \delta_q A \rangle , \quad (4.10)$$

where Π_+ is the projection operator on higher modes. There is in general no reason for the “extra” terms apart from the curvature of the universal bundle sandwiched with zero-modes to vanish, a fact that does not seem to be universally recognized in the earlier physics literature. In fact, we can use the expression for the projector on higher modes,

$$\Pi_+ = D(D^*D)^{-1}D^* , \quad (4.11)$$

to obtain a more explicit expression in terms of the geometrical objects defined above. A crucial ingredient in this calculation is the “lifting” of the complex structures from \mathbb{R}^4 to \mathcal{M}_k of (4.3). The result is

$$R_{mnpq} = \langle \delta_p A, \phi_{mn} \delta_q A \rangle - 4P_{+pq}{}^{rs} \langle \delta_m A, \phi_{nr} \delta_s A \rangle , \quad (4.12)$$

where

$$P_{+pq}{}^{rs} = \frac{1}{4} J^{(a)}_{[p}{}^r J^{(a)}_{q]}{}^s \quad (4.13)$$

($J^{(4)}$ is the identity) is the projection operator on the part of an antisymmetric tensor that commutes with the complex structures, i.e. the $Sp(k)$ part. The first term in (4.12) automatically lies in this subspace. This expression ensures that the curvature on \mathcal{M}_k has the correct holonomy and cyclic symmetry. The projection is a suitable generalization of the selfduality concept in four dimensions. As we will see, an analogous term arises in the effective action for the moduli of $N=2$ super-Yang–Mills with matter.

When we consider fermions coupled to a multi-monopole background [32,24,33], the Dirac equation in four euclidean dimensions reads

$$\begin{bmatrix} 0 & D \\ D^* & 0 \end{bmatrix} \begin{bmatrix} s \\ c \end{bmatrix} = 0 . \quad (4.14)$$

Using the selfduality of the field-strength, one sees that D^*D is the Laplace operator, while DD^* also contains the field-strength as its imaginary part. Therefore, $\ker D \subseteq \ker D^*D = \{0\}$, and the zero-modes must be in the s representation. The index theorem [33] determines their number. The euclidean Dirac equation (4.14) is the linearized time-independent equation of motion for the spinors λ and ψ of the (dimensionally reduced) action (3.1), i.e. it determines their zero-modes. At this level, the distinction between the six-dimensional chiralities is lost. However, the two spinor chiralities couple to A_0 and A_5 with different relative signs (see eq. (2.6)), and this prevents us from making a consistent truncation of the equations of motion to all orders for the time-independent solutions, as is done in [26] for pure $N=2$. This difference will manifest itself in the relative signs of the fermion bilinears of eq. (6.1).

Considering the expression

$$\tilde{\Sigma}_{MN}F^{MN} = -\Gamma_{\mu\nu}F_{\mu\nu} = -\begin{bmatrix} e_\mu e_\nu^* F_{\mu\nu} & 0 \\ 0 & 0 \end{bmatrix} \quad (4.15)$$

occurring in the supersymmetry transformations (3.2), one concludes that the unbroken supersymmetry has transformation parameter $\varepsilon = \begin{bmatrix} 0 \\ \varepsilon_c \end{bmatrix}$.

In the case of adjoint spinors, the Weyl equation is formally identical to the equation for the bosonic moduli, and it is satisfied by

$$s = \delta_m A \lambda^m \quad (4.16)$$

for any set of quaternionic spinors λ^m . As for the bosonic moduli, these are related by the complex structures, so that λ^m can be chosen real (in the case of real fermions). They equal the bosonic moduli in number, i.e. the mode functions span a $4k$ -dimensional vector space over \mathbb{R} . For quantization, it will be essential to use complex fermions, and view it as a $2k$ -dimensional vector space over \mathbb{C} . This is the conventional statement that there are $2k$ (complex) zero-modes in the adjoint representation [32,30,33]. Part of the adjoint zero-modes in $N=2$ super-Yang–Mills (in the $k=1$ sector all of them) are Goldstone fermions, generated by the broken supersymmetry.

For spinors in any representation, the zero-modes, properly normalized, form an orthonormal bundle over \mathcal{M}_k . Fundamental fermions will of course be of special interest – then the number of zero-modes is $2k$ real, i.e. k complex [32,24,33]. We set

$$s = \varrho_\alpha \psi^\alpha, \quad (4.17)$$

where ϱ_α are the quaternionic mode functions. The natural connection on this index bundle is

$$\omega_{m\alpha\beta} = \langle \varrho_\alpha, s_m \varrho_\beta \rangle, \quad (4.18)$$

with ε_m now acting in the appropriate representation of the gauge group, and its curvature can be calculated completely analogously to above:

$$F_{mn\alpha\beta} = \langle \varrho_\alpha, [s_m, s_n] \varrho_\beta \rangle + \langle s_m \varrho_\alpha, \Pi_+ s_n \varrho_\beta \rangle - \langle s_n \varrho_\alpha, \Pi_+ s_m \varrho_\beta \rangle. \quad (4.19)$$

where again the covariant derivative contained in the projection operator (4.11) acts in the appropriate representation.

5. SUPERSYMMETRIC QUANTUM MECHANICS

Semiclassical adiabatic motion of the monopole moduli for supersymmetric Yang–Mills is governed by supersymmetric quantum mechanics, i.e. a one-dimensional supersymmetric sigma model [34–38], with moduli space as target space. On any riemannian manifold, the lagrangian

$$\mathcal{L} = \frac{1}{2} g_{mn}(X) \dot{X}^m \dot{X}^n + \frac{1}{2} g_{mn}(X) \lambda^m D_t \lambda^n \quad (5.1)$$

is supersymmetric. The fermions λ are real, and the covariant derivative is defined as $D_t \lambda^m = \dot{\lambda}^m + \Gamma_{np}^m(X) \dot{X}^n \lambda^p$. The supersymmetry is of the type $N = \frac{1}{2}$, i.e. there is only one, real,

supersymmetry generator. The structure of this model, as well as its generalizations to higher N and number of fermions, is most easily understood in a canonical framework. Due to the second class constraints between the fermions and their momenta, the Dirac procedure yields nonvanishing brackets $\{P_m, \lambda^n\}$ and $\{P_m, P_n\}$. The most natural phase space variable to use instead of the canonical momentum is the velocity $V_m = g_{mn} \dot{X}^n$. One obtains Dirac brackets

$$\begin{aligned}\{V_m, X^n\} &= -\delta_m^n, \\ \{V_m, V_n\} &= \frac{1}{2} R_{mnpq} \lambda^p \lambda^q, \\ \{V_m, \lambda^n\} &= \Gamma_{mp}^n \lambda^p, \\ \{\lambda^m, \lambda^n\} &= g^{mn}.\end{aligned}\tag{5.2}$$

This structure is quite unique – the Jacobi identities demand that g is covariantly constant, that R has the usual expression in terms of Γ and that the Bianchi identities for R are fulfilled. One may as well use an orthonormal basis for the fermions. Then the third equation in (5.2) contains the spin connection instead of the Christoffel connection.

The $N = \frac{1}{2}$ supersymmetry generator is

$$Q = \lambda^m V_m.\tag{5.3}$$

It is not too difficult to see that if there are covariantly constant complex structures, each of these gives another supersymmetry generator $Q_J = \lambda^m J_m{}^n V_n$. In the case of a hyper-Kähler manifold, the number of supersymmetries will be multiplied by four. If the fermions λ are complex, or equivalently, carry an index $i = 1, 2$, the $(N = 1)$ supersymmetry generators are $Q^i = \lambda^{im} V_m$ and anticommute to $\{Q^i, Q^j\} = 2\delta^{ij} \mathcal{H}$, where

$$\mathcal{H} = \frac{1}{2} g^{mn} V_m V_n + \frac{1}{8} R_{mnpq} \lambda^{im} \lambda^{in} \lambda^{jp} \lambda^{jq}.\tag{5.4}$$

The curvature term of course corresponds to a term with opposite sign in \mathcal{L} .

Surprising as it may seem, it is possible to consistently introduce an arbitrary number of fermions into the system (5.2) without destroying the supersymmetry. This is exactly what happens when there are matter multiplets in the supersymmetric field theory. The dynamics of the zero-modes of the matter fermions is completely dictated by supersymmetry. Consider real fermions ψ^α in a bundle over the riemannian manifold, transforming with some connection ω , which in the context of fermion zero-modes will correspond to the connection of the index bundle. With suitable choice of basis, the bundle metric can be chosen as $\delta^{\alpha\beta}$. We then add the brackets

$$\begin{aligned}\{\psi^\alpha, \psi^\beta\} &= \delta^{\alpha\beta}, \\ \{V_m, \psi^\alpha\} &= \omega_m^{\alpha\beta} \psi^\beta,\end{aligned}\tag{5.5}$$

and add to the bracket $\{V_m, V_n\}$ a term $\frac{1}{2} F_{mn\alpha\beta} \psi^\alpha \psi^\beta$, with F the field strength of ω . This is consistent with the Jacobi identities. The $N = \frac{1}{2}$ supersymmetry generator is still given by (5.3), and the hamiltonian obtained as $\frac{1}{2} \{Q, Q\}$ is

$$\mathcal{H} = \frac{1}{2} g^{mn} V_m V_n + \frac{1}{4} F_{mn\alpha\beta} \lambda^m \lambda^n \psi^\alpha \psi^\beta\tag{5.6}$$

(in the case of vanishing ω , the ψ 's are inert under supersymmetry, which is consistent with their equations of motion $\dot{\psi}=0$). The corresponding lagrangian is

$$\mathcal{L} = \frac{1}{2}g_{mn}\dot{X}^m\dot{X}^n + \frac{1}{2}g_{mn}\lambda^m D_t\lambda^n + \frac{1}{2}\psi^\alpha D_t\psi^\alpha - \frac{1}{4}F_{mn\alpha\beta}\lambda^m\lambda^n\psi^\alpha\psi^\beta, \quad (5.7)$$

where $D_t\psi^\alpha = \dot{\psi}^\alpha + \omega_m^{\alpha\beta}\dot{X}^m\psi^\beta$. In the case of N_f fundamental matter multiplets, the ψ 's will come in $2N_f$ copies of an $O(k)$ bundle over \mathcal{M}_k . When target space is hyper-Kähler, it is necessary for F to be selfdual for (5.7) to have extended ($N = \frac{1}{2} \times 4$) supersymmetry. Hitchin has shown (see ref. [24]) that this is indeed the case for the curvature of the index bundle of zero-modes in the fundamental representation. The curvature of a hyper-Kähler manifold is always selfdual (see eq. (4.12)). Selfduality here means that F is a $(1,1)$ -form with respect to any of the complex structures. This definition agrees with the one used earlier when the riemannian curvature of moduli space was calculated.

6. THE ACTION FOR THE MODULI OF N=2 SYM WITH MATTER

In order to make a consistent truncation of the field-theoretic action (3.1), one approximates the motion in a k -monopole background with slow motion in the bosonic and fermionic moduli (zero-modes), a so called collective coordinate expansion [18,17,19]. We expand the equations of motion in $n = \#(\frac{d}{dt}) + \frac{1}{2}\#(\text{fermions})$. At $n=0$ one only has the background fields A with selfdual field-strength. At $n=\frac{1}{2}$, there are the Weyl equations for the upper (s chirality) components of λ and ψ , which we denote α and β , respectively. Their lower (c chirality) components vanish to this order. The time dependence of the bosonic moduli is modeled so that $A(x, t) = A(x, X(t))$. Then the equations at order $n=1$ imply, using $\dot{A} = \dot{X}^m(\delta_m A + D\varepsilon_m)$,

$$\begin{aligned} A_0 &= \dot{X}^m\varepsilon_m + (D^*D)^{-1}(-\alpha^*\alpha + \frac{1}{2}\beta_f^* \times \beta_f), \\ A_5 &= (D^*D)^{-1}(\alpha^*\alpha + \frac{1}{2}\beta_f^* \times \beta_f), \\ q_f^* &= -(D^*D)^{-1}(\alpha^*\beta_f). \end{aligned} \quad (6.1)$$

Inserting these solutions into the action (3.1) and discarding terms with $n > 2$ gives the action (5.7), after integrating over three-space and using the expressions for connections and curvatures on moduli space. There is also a constant topological term $-4\pi k$, so the supersymmetry algebra contains a central charge at the classical level. Since moduli space is hyper-Kähler, the action possesses $N = \frac{1}{2} \times 4$ supersymmetry. We are amused to note that the second term in (4.19) is present, and arises exactly from interaction with the bosonic matter field q . In order to see this, one lets s_m act on the Weyl equation to obtain $\delta_m A^* \varrho_\alpha = -D^* s_m \varrho_\alpha$ and uses the expression for the higher mode projector (4.11) in the solution (6.1) for q . Then, $Dq^* = \lambda^m \psi^\alpha \Pi_{+s_m} \varrho_\alpha$. A non-supersymmetric theory without matter bosons would not get the correct curvature term for the fermionic moduli. The result is of course valid also for $N=4$ (which is $N=2$ with one adjoint matter multiplet), where earlier calculations [25] seem to have discarded the contribution from q in the effective action and approximated the curvature with its first term in (4.10).

7. ON THE QUANTIZATION OF THE EFFECTIVE ACTION

One would like to use the low energy effective action (5.7) in order to draw conclusions about the selfduality of the $N=2$ theory with four fundamental hypermultiplets. The details of this are left to a forthcoming publication, but in order to put the present work in some perspective, we would like to indicate the route to take.

The task is to find BPS-saturated states in the monopole spectrum. The BPS limit for the mass is already present as the topological term $4\pi k$ in the hamiltonian obtained from the field theory, so one searches for zero energy states of the hamiltonian (5.6). The zero-modes of the adjoint fermions are divided into creation and annihilation operators using the Kähler property of moduli space, i.e. by identifying one of the complex structures with i . Then, in the case of pure $N=2$, the states may be identified with antiholomorphic forms in a Dolbeault complex [35,38,39,27], and there is a nice correspondence with the cohomology of the manifold (which is severely restricted by the hyper-Kähler property).

When the fundamental fermions and the field strength F are present, the situation changes slightly. It is probably advantageous to treat the fundamental fermions with gamma matrix quantization [2], analogous to the way the fermions in the Ramond sector of the superstring are treated. Then the forms carry a $Spin(2kN_f)$ spinor index. The direct identification of supercharges with exterior derivatives no longer holds, since the fundamental fermions are affected by supersymmetry. Instead, there will be an extra contribution behaving as a connection term. We envisage that the zero energy states correspond to forms which are harmonic with respect to the arising covariant laplacian. The selfduality of F should play an important rôle here.

In reference [2] part of the supposed mechanism behind the expected selfduality of the model in question has been briefly sketched. An $Sl(2, \mathbb{Z})$ transformation on the coupling constant and θ angle has to be accompanied by a $Spin(8)$ triality rotation (in the $k=1$ sector) to obtain a mapping between dual states. One also has to remember that the masses induced by the Higgs mechanism for the fundamental fields of the theory are different in the adjoint and fundamental representation (since the BPS bound relates mass to charge). This means that duality will map the different multiplets in (3.1) into sectors with different magnetic charge k . We believe that a careful treatment along the lines sketched here will verify these claims. This work is in progress.

Note added: Shortly after the completion of this work two papers appeared [40,41] that essentially depart from the effective action derived in this paper and find the BPS-saturated spectrum for magnetic charges $k=1, 2$ along the lines sketched in section 7. Their results support the selfduality hypothesis for the $N=2$ model with four fundamental hypermultiplets.

REFERENCES

- [1] N. Seiberg and E. Witten, *Nucl.Phys.* **B426** (1994) 19; erratum: *ibid.* **B430** (1994) 485 ([hep-th/9407087](#)).
- [2] N. Seiberg and E. Witten, *Nucl.Phys.* **B431** (1994) 484 ([hep-th/9408099](#)).
- [3] C. Vafa and E. Witten, *Nucl.Phys.* **B431** (1994) 3 ([hep-th/9408074](#)).
- [4] K. Intriligator and N. Seiberg *Nucl.Phys.* **B431** (1994) 551 ([hep-th/9408155](#)).
- [5] C. Montonen and D. Olive, *Phys.Lett.* **72B** (1977) 117.
- [6] A. Font, L.E. Ibáñez, D. Lüst and F. Quevedo, *Phys.Lett.* **249B** (1990) 35.
- [7] S. Mandelstam, *Nucl.Phys.* **B213** (1983) 149.
- [8] L. Brink, O. Lindgren and B.E.W. Nilsson, *Phys.Lett.* **123B** (1983) 323.
- [9] M. Sohnius and P. West, *Phys.Lett.* **100B** (1981) 45.
- [10] P.S. Howe, K.S. Stelle and P.C. West, *Phys.Lett.* **124B** (1983) 55.
- [11] P.S. Howe, K.S. Stelle and P.K. Townsend, *Nucl.Phys.* **B214** (1983) 519.
- [12] O. Piguet and K. Sibold, *Helv.Phys.Acta* **63** (1990) 71.
- [13] N. Seiberg *Phys.Lett.* **206B** (1988) 75.
- [14] H. Osborn, *Phys.Lett.* **83B** (1979) 321.
- [15] E.B. Bogomolny, *Sov.J.Nucl.Phys.* **24** (1976) 449.
- [16] M.K. Prasad and C.M. Sommerfield, *Phys.Rev.Lett.* **35** (1975) 760.
- [17] N.S. Manton *Phys.Lett.* **110B** (1982) 54.
- [18] N.H. Christ and T.D. Lee, *Phys.Rev.* **D22** (1980) 939.
- [19] R. Rajaraman, “*Solitons and Instantons*”, North-Holland (1982).
- [20] J. Wess and J. Bagger, “*Supersymmetry and Supergravity*”, Princeton Univ. Press (1983).
- [21] T. Kugo and P. Townsend, *Nucl.Phys.* **B221** (1983) 357.
- [22] A. Sudbery, *J.Phys.* **A17** (1984) 939.
- [23] M.F. Atiyah and I.M. Singer, *Proc.Nat.Acad.Sci.USA* **81** (1984) 2597.
- [24] N.S. Manton and B.J. Schroers, *Ann.Phys.* **225** (1993) 290.
- [25] J. Blum, *Phys.Lett.* **333B** (1994) 92 ([hep-th/9401133](#)).
- [26] J.P. Gauntlett, *Nucl.Phys.* **B411** (1994) 443 ([hep-th/9305068](#)).
- [27] A. Sen, *Phys.Lett.* **B329** (1994) 217 ([hep-th/9402032](#)).
- [28] A. Parkes and P.C. West, *Phys.Lett.* **122B** (1983) 365.
- [29] A. D’Adda, R. Horsley and P. Di Vecchia, *Phys.Lett.* **76B** (1978) 298.
- [30] M.F. Atiyah and N. Hitchin, “*The Geometry and Dynamics of Magnetic Monopoles*”, Princeton University Press, Princeton (NJ) (1988).
- [31] J.A. Harvey and A. Strominger, *Commun.Math.Phys.* **151** (1993) 221 ([hep-th/9108020](#)).
- [32] R. Jackiw and C. Rebbi, *Phys.Rev.* **D13** (1976) 3398.
- [33] C. Callias, *Commun.Math.Phys.* **62** (1978) 213.
- [34] E. Witten, *Phys.Rev.* **D16** (1977) 2991.
- [35] E. Witten, *Nucl.Phys.* **B202** (1982) 253.
- [36] P. Di Vecchia and S. Ferrara, *Nucl.Phys.* **B130** (1977) 93.
- [37] J. Bagger, Lectures at Bonn-NATO Adv. Study Inst. on Supersymmetry, 1984, *Bonn Supersym.ASI* (1984) 45.
- [38] L. Alvarez-Gaumé, *Commun.Math.Phys.* **90** (1983) 161.
- [39] J.P. Gauntlett, *Nucl.Phys.* **B400** (1993) 103 ([hep-th/9205008](#)).
- [40] S. Sethi, M. Stern and E. Zaslow, [hep-th/9508117](#).
- [41] J.P. Gauntlett and J.A. Harvey, [hep-th/9508156](#).